The Mayer-Vietoris Pyramid

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References

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- **Zigzag persistent homology and real-valued functions.** Gunnar Carlsson, Vin de Silva, Dmitriy Morozov. 2009.
- **Zigzag Persistence.** Gunnar Carlsson, Vin de Silva. 2008.
- **Extending Persistence Using Poincare and Lefschetz Duality.** David Cohen-Steiner, Herbert Edelsbrunner, John Harer. 2009.
- **Algebraic Topology.** Allen Hatcher. 2000.

Mayer-Vietoris Pyramid

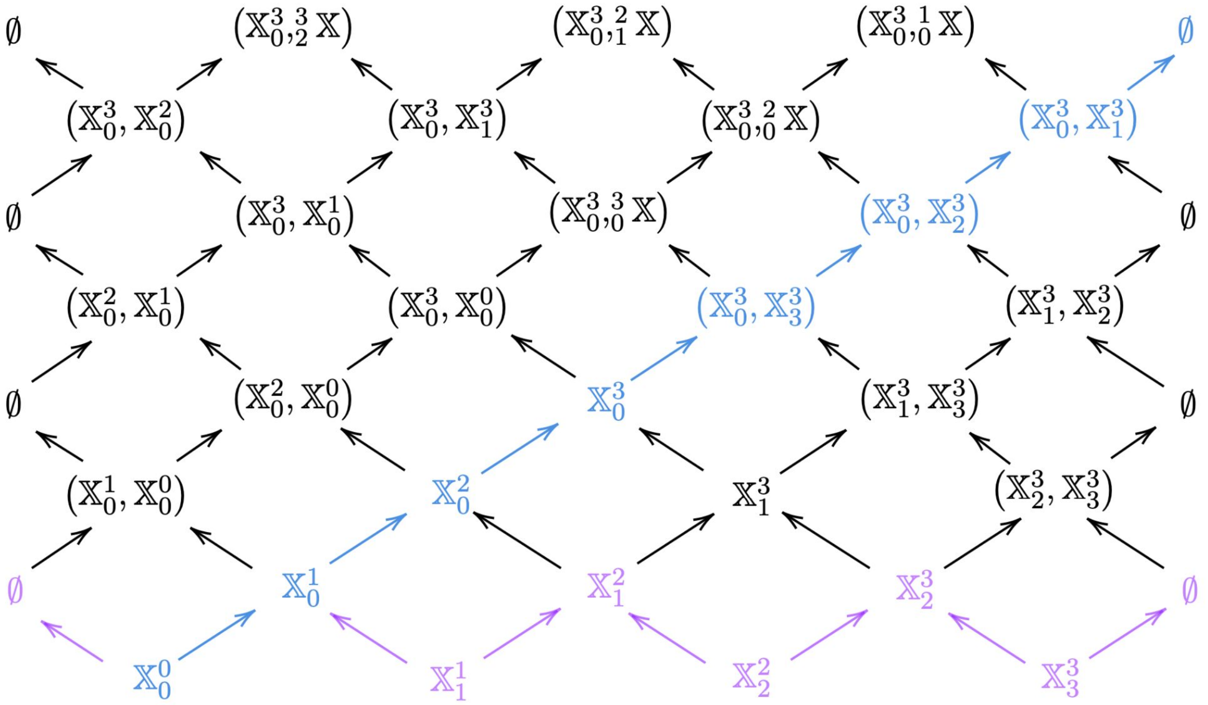


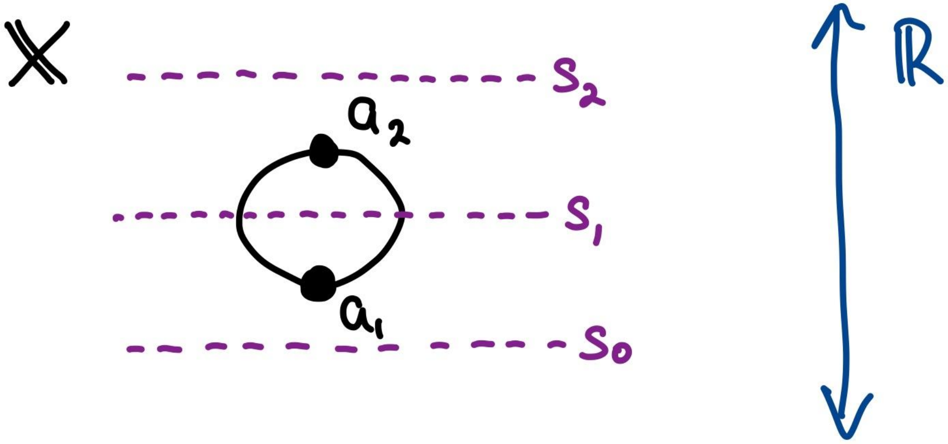
Figure 4: Pyramid for the case $n = 3$, with ${}_i^j\mathbb{X} := \mathbb{X}_0^i \cup \mathbb{X}_j^n$.

- $\mathbb{X}$ is a topological space
- $f : \mathbb{X} \to \mathbb{R}$ is continuous
- $(\mathbb{X}, f)$ is of Morse type with critical values

  $a_1 < a_2 < \cdots < a_n$

- select s_i such that

$$-\infty < s_0 < a_1 < s_1 < a_2 < \cdots < s_{n-1} < a_n < s_n < \infty$$



- $\mathbb{X}_i^j = f^{-1}([s_i, s_j])$ called interlevelsets

$$
\begin{tikzcd}
 & (\mathbb{X}_0^3, \mathbb{X}_0^0) & \\
(\mathbb{X}_0^2, \mathbb{X}_0^0) \arrow[ur] & & (\mathbb{X}_0^3, \emptyset) \arrow[ul] \\
 & (\mathbb{X}_0^2, \emptyset) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P_1 \cup P_2 & \\
P_1 \arrow[ur, hook] & & P_2 \arrow[ul, hook'] \\
 & P_1 \cap P_2 \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}
$$

$$P_1 = (\mathbb{X}_0^2, \mathbb{X}_0^0)$$
$$P_2 = (\mathbb{X}_0^3, \emptyset)$$

- applying homology gives an exact square

$$\begin{tikzcd}
 & H_p(A_1 \cup A_2) & \\
H_p(A_2) \arrow[ur] & & H_p(A_1) \arrow[ul] \\
 & H_p(A_1 \cap A_2) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Definition 3.1 (Exact Square). *An* exact square *is a diagram of vector spaces*

$$\begin{tikzcd} V_3 \arrow[r, "g_2"] & V_4 \\ V_1 \arrow[u, "f_2"] \arrow[r, "f_1"] & V_2 \arrow[u, "g_1"'] \end{tikzcd}$$

that satisfies the condition $\mathrm{Ker}(V_2 \oplus V_3 \to V_4) = \mathrm{Im}(V_1 \to V_2 \oplus V_3)$ *in the sequence*

$$V_1 \longrightarrow V_2 \oplus V_3 \longrightarrow V_4,$$

where $(V_1 \to V_2 \oplus V_3) = f_1 \oplus f_2$ *and* $(V_2 \oplus V_3 \to V_4) = g_1 - g_2$.

- Strong Diamond Principle

$$A_\cup : \cdots A_{k-1} \hookrightarrow A_{k-1} \cup A_{k+1} \hookleftarrow A_{k+1} \cdots$$

$$A_\cap : \cdots A_{k-1} \hookleftarrow A_{k-1} \cap A_{k+1} \hookrightarrow A_{k+1} \cdots$$

$$\mathbb{V}^+ = H_*(A_\cup)$$

$$\mathbb{V}^- = H_*(A_\cap)$$

$$\Rightarrow \mathbb{B}(\mathbb{V}^+) \simeq \mathbb{B}(\mathbb{V}^-)$$

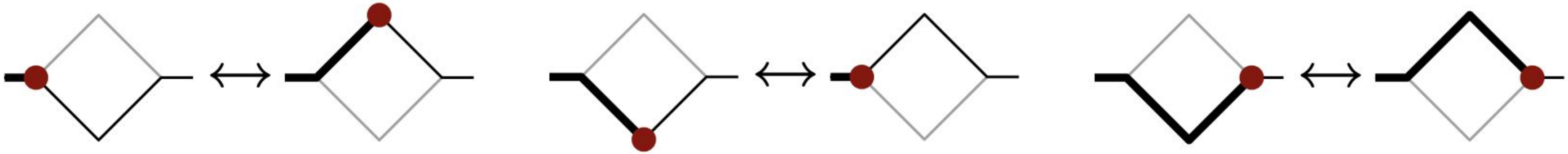
- the Strong Diamond Principle applies to all diamonds in a Mayer-Vietoris Pyramid
- Pyramid contains zigzag modules for
  - levelsets zigzag pers.
  - extended pers.
- we can incrementally "step" one module into the other

Theorem 3.7. *([11, Pyramid Theorem]) There is an explicit bijection between the extended persistence barcode and the levelsets zigzag persistence barcode of* $(\mathbb{X}, f)$*, that respects homological dimension except for possible shifts of degree* $\mathrm{d} \in \{-1, 1\}$.

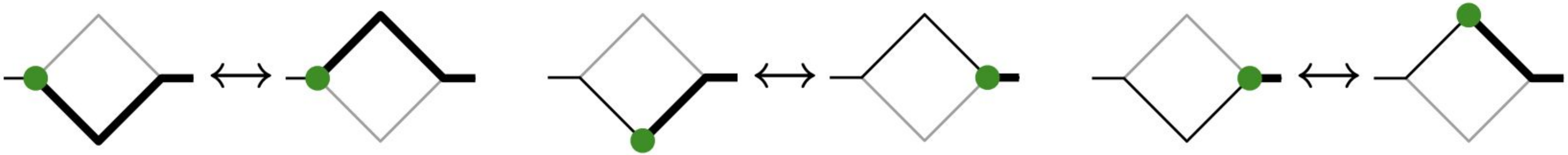
Theorem 3.9 (Barcode Bijection). *One has the following correspondence between the intervals of the extended persistence barcode (left) and intervals of the levelsets zigzag persistence barcode (right).*

Type	*Extended*	*Levelsets zigzag*
$I\ (i<j)$	$[\mathbb{X}_0^i, \mathbb{X}_0^{j-1}]$	$[\mathbb{X}_{i-1}^i, \mathbb{X}_{j-1}^{j-1}]$
$II\ (i<j)$	$[(\mathbb{X}_0^n, \mathbb{X}_{j-1}^n), (\mathbb{X}_0^n, \mathbb{X}_i^n)]^+$	$[\mathbb{X}_i^i, \mathbb{X}_{j-1}^j]$
$III\ (i \leq j)$	$[\mathbb{X}_0^i, (\mathbb{X}_0^n, \mathbb{X}_j^n)]$	$[\mathbb{X}_{i-1}^i, \mathbb{X}_{j-1}^j]$
$IV\ (i<j)$	$[\mathbb{X}_0^j, (\mathbb{X}_0^n, \mathbb{X}_i^n)]^+$	$[\mathbb{X}_i^i, \mathbb{X}_{j-1}^{j-1}]$

Birth Transformations:



Death Transformations:



- a quiver is an oriented graph
- a quiver is of type A when there is one edge between any two vertices
- a representation of a quiver replaces the vertices with vector spaces
- an interval representation (barcode)

$$\mathbb{I}(1,3) = 0 \xrightarrow{0} k \xleftarrow{id} k \xrightarrow{id} k \xleftarrow{0} 0$$

- representations of type A quivers
  - zigzag levelsets persistence module
  - extended persistence module

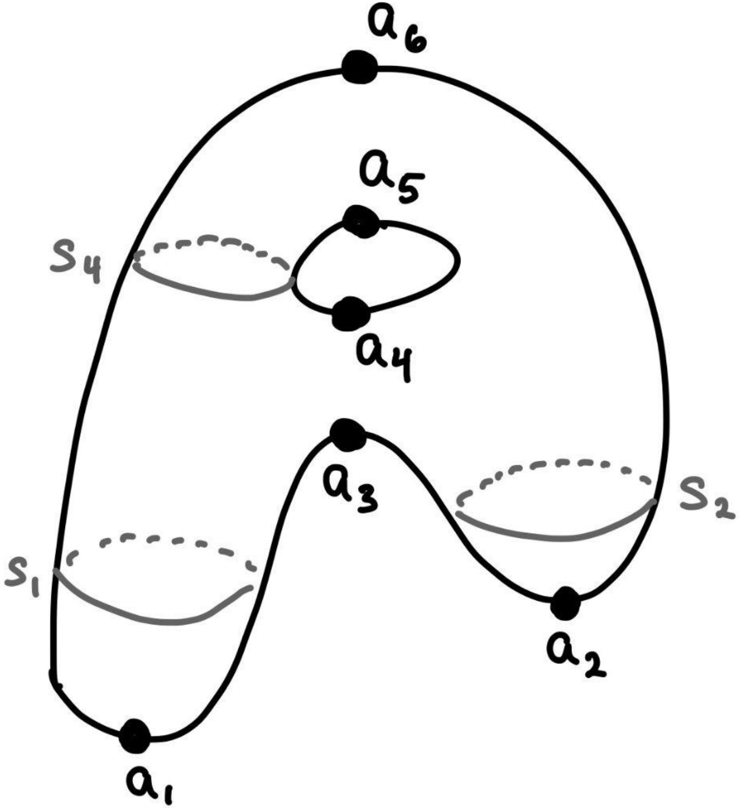
- by Gabriel's Theorem each such representation $\mathbb{V}$ admits a barcode

$$\mathbb{V} \simeq \mathbb{I}(b_1, d_1) \oplus \cdots \oplus \mathbb{I}(b_n, d_n)$$

$$-\infty < s_0 < a_1 < s_1 < a_2 < \cdots < s_{n-1} < a_n < s_n < \infty$$

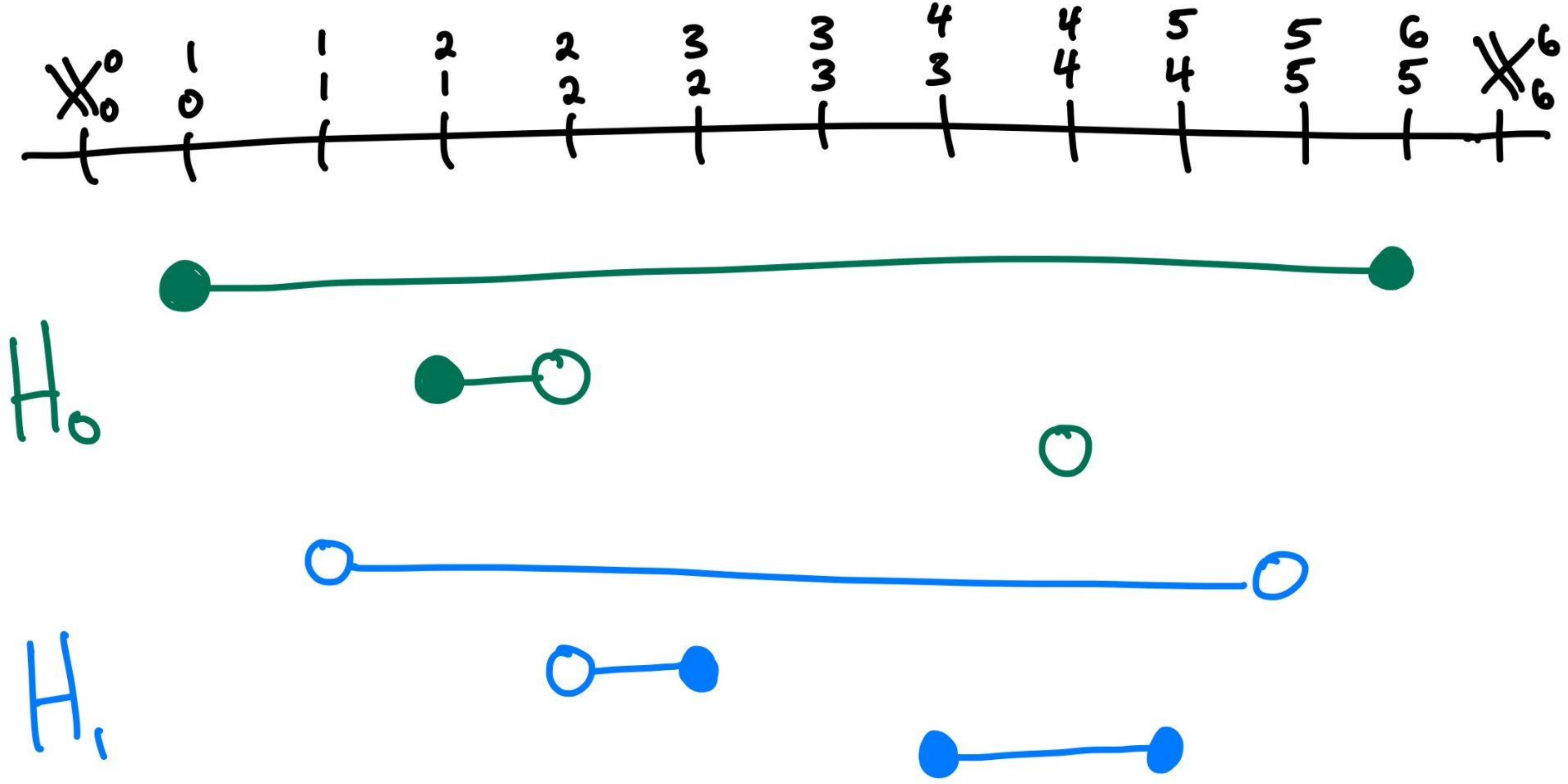
- persistence interval labelling
  - $\mathbb{X}_i^i \longleftrightarrow (a_i, a_{i+1}) \longleftrightarrow (b, d)$
  - $\mathbb{X}_{i-1}^i \longleftrightarrow [a_i, a_i] \longleftrightarrow [b, d]$

- examples
  - $[\mathbb{X}_2^3, \mathbb{X}_5^5] \longleftrightarrow [a_3, a_6)$
  - $[\mathbb{X}_5^5, \mathbb{X}_7^8] \longleftrightarrow (a_5, a_8)$



$$\mathbb{X}_i^i \longleftrightarrow (a_i, a_{i+1})$$

$$\mathbb{X}_{i-1}^i \longleftrightarrow [a_i, a_i]$$


$\mathbb{X}^0_0$
$^1_0$
$^1_1$
$^2_1$
$^2_2$
$^3_2$
$^3_3$
$^4_3$
$^4_4$
$^5_4$
$^5_5$
$^6_5$
$\mathbb{X}^6_6$
H_0
H_1